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APPROXIMATE AUTOMORPHISMS OF SOLVABLE LINEAR CONNECTED LIE GROUPS

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Abstract. We apply well-known results concerning quasirepresentations and pure pseudorepresentations to study approximate automorphisms of solvable connected linear Lie groups with small defect.

§ 1. Introduction

In some cases, an information concerning properties of mappings of groups that are close to representations can help to study mappings close to homomorphisms with more complicated images. Here we discuss mappings of rather simple groups that are close to automorphisms of the groups.

The scheme of consideration uses the construction of a quasirepresentation closely related to the given approximate automorphism and in the subsequent proof of the fact that the image of the "corrected" mapping, which is already an ordinary representation, coincides with the image of some chosen faithful representation of the group. This way is efficient, in contrast to many unhappy attempts to prove, in the unbounded situation, that two sufficiently close representations are equivalent by an operator close to the identity mapping of the image of the chose representation onto itself.

The main result of the present paper claims that every approximate automorphism, with sufficiently small defect, of a solvable connected linear Lie group is close to an ordinary automorphism of the group.

Recall the following fact about the linearity of connected Lie groups. Let G be a connected Lie group. As was proved in [1] (see also comments in [2]),

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a connected Lie group G is linear if and only if G admits a faithful uniformly continuous representation and if and only if G embeds continuously into the invertible group of a unital Banach algebra. This gives a clear framework for the problem under consideration.

Here and below, we use the definitions, notions, and notation of [3] without additional comments.

§ 2. Main Theorem

Theorem. *Every approximate automorphism, with sufficiently small defect, of a solvable connected linear Lie group is close to an ordinary automorphism of the group, where the closeness depends on the defect only.*

Proof. Let G be a solvable connected linear Lie group. Let π be a faithful linear representation of G, and let $\pi\colon G \to \mathrm{GL}_n(\mathbb{R})$ be a faithful finite-dimensional linear representation of G in a normed n-dimensional space E_n. Let α be an approximate automorphism of G, i.e.,

$$\|\alpha(gh) - \alpha(g)\alpha(h)\|_{\mathcal{L}(E_n)} \le \varepsilon \quad \text{for all} \quad g, h \in G$$

$$\text{and some sufficiently small} \quad \varepsilon.$$

Since the family $\alpha(G)$ is solvable by assumption, it follows that $\pi(\alpha(G))$ is a solvable subgroup of the linear group $\mathrm{GL}_n(\mathbb{R})$. By the Lie theorem, this group can be realized (in some basis) in an upper triangular form. Since G is solvable, it is amenable, and therefore the quasirepresentation $\pi \circ \alpha$ admits a close ordinary representation [3], which we denote by ρ. It is clear that the diagonal elements of our upper triangular realization define a quasirepresentation of G with the same small defect. Since G is solvable, it is amenable, and therefore the diagonal quasirepresentation admits a close ordinary representation ϱ, which is also diagonal.

Since α is an approximate automorphism with small defect, and close characters of a commutative Lie group are equal, we see that, for sufficiently small defect, ϱ is a permutation of the original diagonal.

Further, it is clear from the Lie algebra point of view that the restriction of the quasirepresentation ρ (well approximating $\pi \circ \alpha$) to the largest unipotent subgroup U of G (in the chosen realization) has the matrix entries which are quasipolynomials in the Lie algebra coordinates of U without the constant term, and uniformly close quasipolynomials of this kind are equal.

The general form of the approximating representation [3] is

$$\rho(g) = \begin{pmatrix} \iota(g) & \varphi(g) & \sigma(g) & \tau(g) \\ 0 & \beta(g) & 0 & \theta(g) \\ 0 & 0 & \gamma(g) & \chi(g) \\ 0 & 0 & 0 & \delta(g) \end{pmatrix}, \qquad g \in G.$$

Here $t_{23}(g) = 0$, because the corresponding subspace is invariant with respect to ρ, the mappings ι, δ, γ, σ, and χ are bounded; the mappings t_1 and t_2 given by

$$t_1(g) = \begin{pmatrix} \iota(g) & \varphi(g) \\ 0 & \beta(g) \end{pmatrix}$$

and

$$t_2(g) = \begin{pmatrix} \beta(g) & \theta(g) \\ 0 & \delta(g) \end{pmatrix},$$

are representations of G; the mapping τ is a quasicocycle with respect to t_1 and t_2, i.e., the mapping

$$(g,h) \mapsto \tau(gh) - \alpha(g)\tau(h) - \varphi(g)\rho(h) - \tau(g)\delta(h),$$

$g, h \in G$, is bounded. This shows that, in the case under consideration, ι, γ, and δ are direct sums of characters (because these are bounded representations of solvable Lie groups), and admit no small deformation, τ is diagonal and has either zeros or linear factors on the diagonal, which can be seen from the definition of subspaces decomposing E_n (see [3]), and hence τ defines these linear factors uniquely (without any influence of a small perturbation), φ and χ are trivial, and the unbounded representation β (whose contragredient representation is also unbounded) is equivalent to any representation obtained under a uniformly small perturbation.

Therefore, after a correct re-indexing of equal diagonal characters and correct choice of bases in the corresponding eigensubspaces, we see that the representation ρ is equivalent to the composition of some automorphism β of G and the representation α, as was to be proved.

§ 3. Discussion

This consideration gives hope that a similar treatment can be made for approximate automorphisms of simple Lie groups that are not Hermitian symmetric [4].

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